Parametric Equations of Ellipses and Hyperbolas

It is often useful to find parametric equations for conic sections. In particular, there are standard methods for finding parametric equations of ellipses and hyperbolas.

Parametric Equations of Ellipses

An ellipse centered at the origin has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

As shown in example 7 of section 9.7, a simple way to parametric the ellipse is to use the equations $x = a\cos t$, $y = b\sin t$. This is because if you plug these expressions into the above equation, then the equation reduces to the trigonometric identity $\cos^2 t + \sin^2 t = 1$.

Generalizations: If the center of the ellipse is at (h, k), then the equation is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

In this case, the parametric equations $x = h + a\cos t$, $y = k + b\sin t$ will work (again, you can just plug these expressions into the last equation above to obtain the identity $\cos^2 t + \sin^2 t = 1$).

Finally, $x = h + a\cos\omega t$, $y = k + b\sin\omega t$. will also parametrize an ellipse centered at the (h, k). The factor ω will determine the speed and direction that the curve is traced out as t increases (see example 7). You can also switch cos and sin and still get an ellipse, but this will make the starting point different (i.e., the point corresponding to $t = 0$).

Parametric Equations of Hyperbolas

A hyperbola centered at the origin with a left-right opening has the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

In this case, we can use the equations $x = a\sec t$, $y = b\tan t$. This is because if you plug these expressions into the above equation, then the equation reduces to the trigonometric identity $\sec^2 t - \tan^2 t = 1$.

Likewise, a hyperbola centered at the origin with an up-down opening has the equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

In this case, you have to switch the expressions for x and y, using the equations $x = b\tan t$, $y = a\sec t$.

Generalizations: If the center of the hyperbola is at (h, k), then the equation is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

or

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

In the first case, the parametric equations $x = h + a\sec t$, $y = k + b\tan t$ will work (again, you can just plug these expressions into the next-to-last equation above to obtain the identity $\sec^2 t - \tan^2 t = 1$), and in the second case it will be $x = h + b\tan t$, $y = k + a\sec t$.

Finally, just as in the case of the ellipse, you can replace t by ωt and still get a parametrization of the hyperbola. The factor ω will determine the speed and direction that the curve is traced out as t increases.