Math 142
Fall 2000

Rotation of Axes

In section 11.4, we found that every equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

with A and C not both 0, can be transformed by completing the square into a standard equation of a translated conic section. Thus, the graph of this equation is either a parabola, ellipse, or hyperbola with axes parallel to the x and y-axes (there is also the possibility that there is no graph or the graph is a "degenerate" conic: a point, a line, or a pair of lines). In this section, we will discuss the equation of a conic section which is rotated by an angle θ, so the axes are no longer parallel to the x and y-axes.

To study rotated conics, we first look at the relationship between the axes of the conic and the x and y-axes. If we label the axes of the conic by u and v, then we can describe the conic in the usual way in the uv-coordinate system with an equation of the form

$$au^2 + cv^2 + du + ev + f = 0 \tag{2}$$

Label the axis in the first quadrant by u, so that the u-axis is obtained by rotating the x-axis through an angle θ, with $0 < \theta < \pi/2$ (see Figure 1). Likewise, the v-axis is obtained by rotating the y-axis through the angle θ. Now if P is any point with usual polar coordinates (r, α), then the polar coordinates of P in the uv system are $(r, \alpha - \theta)$ (again, see Figure 1). Thus,

$$\begin{aligned} u &= r\cos(\alpha - \theta) = r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) \\ &= r(\cos\alpha)\cos\theta + (r\sin\alpha)\sin\theta = x\cos\theta + y\sin\theta \\ &\text{and} \\ v &= r\sin(\alpha - \theta) = r(\sin\alpha\cos\theta - \cos\alpha\sin\theta) \\ &= r(\sin\alpha)\cos\theta - (r\cos\alpha)\sin\theta = y\cos\theta - x\sin\theta \end{aligned} \tag{3}$$

These two equations in (3) describe how to find the uv-coordinates of a point from its xy-coordinates.

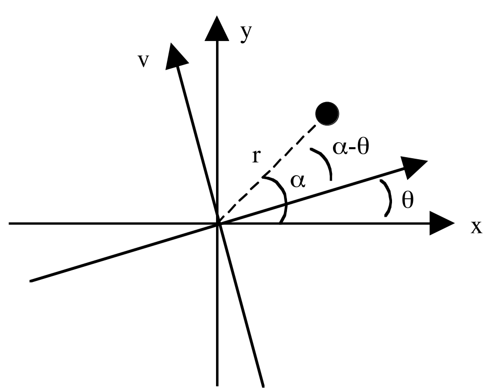

Figure 1

Using a similar derivation, if we start with the uv-coordinates, then the xy-coordinates are given by the equations

$$(4) \qquad x = u\cos\theta - v\sin\theta \qquad \text{and} \qquad y = v\cos\theta + u\sin\theta$$

Now if we substitute the expressions for u and v from (3) into equation (2), we will end up with an equation of the form

$$(5) \qquad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

In other words, a "mixed" xy term has been introduced.

Example 1. Starting with the ellipse $4u^2+9v^2 = 36$ in the uv-system, we obtain equation (5) in the xy-system as follows:

$$\begin{aligned}
&4(x\cos\theta + y\sin\theta)^2 + 9(y\cos\theta - x\sin\theta)^2 = 36\\
&\Rightarrow 4(x^2\cos^2\theta + 2xy\cos\theta\sin\theta + y^2\sin^2\theta) + 9(y^2\cos^2\theta - 2xy\cos\theta\sin\theta + x^2\sin^2\theta) = 36\\
&\Rightarrow (4\cos^2\theta + 9\sin^2\theta)x^2 - (10\cos\theta\sin\theta)xy + (4\sin^2\theta + 9\cos^2\theta)y^2 = 36
\end{aligned}$$

If we know that the uv-system is obtained from the xy-system by a rotation of $\theta = \frac{\pi}{3}$ radians, for example, then $\cos\theta = \frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$, and the final equation is then $\frac{31}{4}x^2 - \frac{5\sqrt{3}}{2}xy + \frac{21}{4}y^2 = 36$.

Conversely, we can show that every equation of the form (5) represents a conic section. To do this, we show that this equation is really just the equation of a rotated conic. In other words, we want to apply the conversion formulas (4) for a suitable angle θ so that the new uv equation has the form (2).

First, we can assume that $B \neq 0$, for otherwise there is nothing to do. Now substitute formulas (4) into equation (5):

$$\begin{aligned}
A(u\cos\theta - v\sin\theta)^2 + B(u\cos\theta - v\sin\theta)(v\cos\theta + u\sin\theta) + C(v\cos\theta + u\sin\theta)^2 +\\
D(u\cos\theta - v\sin\theta) + E(v\cos\theta + u\sin\theta) + F = 0
\end{aligned}$$

Expanding this expression and collecting like terms yields the equation

$$au^2 + buv + cv^2 + du + ev + f = 0$$

with

$$\begin{aligned}
a &= A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta\\
b &= -2A\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) + 2C\sin\theta\cos\theta\\
c &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta\\
d &= D\cos\theta + E\sin\theta\\
e &= -D\sin\theta + E\cos\theta\\
f &= F
\end{aligned}$$

Our goal is for b to be equal to 0. Using the double angle formulas

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \text{and} \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

we obtain

$$b = B\cos 2\theta + (C - A)\sin 2\theta$$

and it follows that $b = 0$ if

$$\cot 2\theta = \frac{A - C}{B} \qquad (6)$$

Since $B \neq 0$, this equation can always be solved, and θ can be chosen so that $0 < \theta < \pi/2$.

Example 2. Consider the equation $x^2 + 2xy + y^2 + 4\sqrt{2}x - 4\sqrt{2}y = 0$. $A = 1$, $C = 1$, and $B = 2$, so formula (6) implies that $\cot 2\theta = 0$. Therefore $\cos 2\theta = 0$, so $2\theta = \pi/2$ (remember that $0 < \theta < \pi/2$, so $0 < 2\theta < \pi$) and thus $\theta = \pi/4$. It then follows from (4) that $x = u/\sqrt{2} - v/\sqrt{2}$ and $y = v/\sqrt{2} + u/\sqrt{2}$. Substituting these values into the given equation and then simplifying the result eventually yields $2u^2 - 8v = 0$, or $u^2 = 4v$. This is a parabola with vertex at the origin, axis $u = 0$ (i.e., the v-axis), focus $(u, v) = (0, 1)$, and directrix $v = -1$. Now using formulas (4), it follows that the xy-coordinates of the focus are $(x, y) = (-1/\sqrt{2}, 1/\sqrt{2})$, the axis is $y = -x$ (since $\theta = \pi/4$, the v-axis is the line through the origin with slope equal to $\tan(3\pi/4) = -1$), and the directrix is the line $y = x - \sqrt{2}$ (the line contains the point $(u, v) = (0, -1)$, which has xy-coordinates $(x, y) = (1/\sqrt{2}, -1/\sqrt{2})$, and the slope is 1 since it is parallel to the u-axis (which has slope $\tan\theta = \tan(\pi/4) = 1$)). See Figure 2.

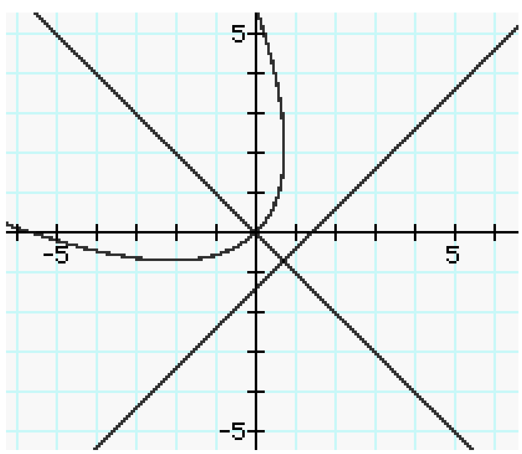

Figure 2

Example 3. Consider the equation $5x^2 + 4\sqrt{3}xy + 9y^2 + 3\sqrt{3}x - 3y = 30$. $A = 5$, $C = 9$, and $B = 4\sqrt{3}$, so formula (6) implies that $\cot 2\theta = -1/\sqrt{3}$. Therefore $\tan 2\theta = -\sqrt{3}$, so $2\theta = 2\pi/3$ (remember that $0 < \theta < \pi/2$, so $0 < 2\theta < \pi$) and thus $\theta = \pi/3$. It then follows from (4) that $x = u/2 - \sqrt{3}v/2$ and $y = v/2 + \sqrt{3}u/2$. Substituting these values into the given equation and then simplifying the result eventually yields $11u^2 + 3v^2 - 6v = 30$. Completing the square, we obtain $11u^2 + 3(v - 1)^2 = 33$. This is an ellipse with center $(u, v) = (0, 1)$, major axis on the line $u = 0$, and minor axis on the line $v = 1$. The length of the major axis is $2\sqrt{11}$ and the length of the minor axis is $2\sqrt{3}$. Thus, the vertices are

$(u,v) = (0, 1 \pm \sqrt{11})$, and the foci are $(u,v) = (0, 1 \pm \sqrt{8})$. To figure out the xy-coordinates, first note (using formulas (4)) that the center has xy-coordinates $(x,y) = (-\sqrt{3}/2, 1/2)$. If follows that the minor axis is on the line $y = \sqrt{3}x + 2$ (the slope is $\tan\theta = \tan(\pi/3) = \sqrt{3}$, and it must pass through the center) and the major axis is on the line $y = -x/\sqrt{3}$ (the slope is $-1\sqrt{3}$ since it is perpendicular to the minor axis, and it must also pass through the center). Using formulas (4) again, the xy-coordinates of the vertices are $(\frac{-\sqrt{3}\pm\sqrt{33}}{2}, \frac{1\pm\sqrt{11}}{2})$, and the foci are $(\frac{-\sqrt{3}\pm\sqrt{24}}{2}, \frac{1\pm\sqrt{8}}{2})$. See Figure 3.

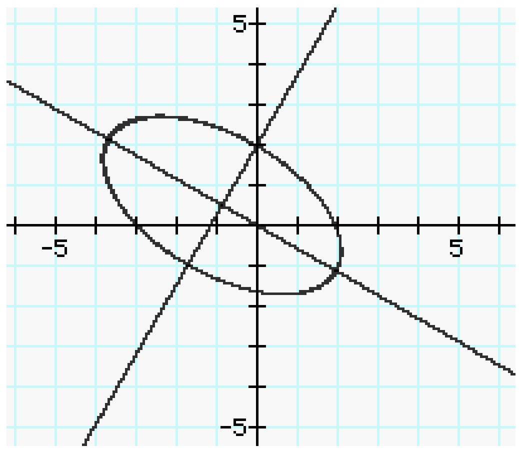

Figure 3

Example 4. Consider the equation $x^2 + 3xy - 2y^2 - 5 = 0$. This example is a little more difficult. $A = 1$, $C = -2$, and $B = 3$, so formula (6) implies that $\cot 2\theta = 1$. Therefore $\tan 2\theta = 1$, so $2\theta = \pi/4$ (remember that $0 < \theta < \pi/2$, so $0 < 2\theta < \pi$), and thus $\theta = \pi/8$. It follows from (4) that $x = cu - sv$ and $y = cv + su$, where $c = \cos(\pi/8) = \frac{\sqrt{2+\sqrt{2}}}{2}$ and $s = \sin(\pi/8) = \frac{\sqrt{2-\sqrt{2}}}{2}$ (using the half-angle formulas). Substituting these values into the given equation and then simplifying the result eventually yields $\frac{u^2}{a^2} - \frac{v^2}{b^2} = 1$, where $a^2 = \frac{10}{3\sqrt{2}-1}$ and $b^2 = \frac{10}{3\sqrt{2}+1}$, so $a \approx 1.76$ and $b \approx 1.38$. This is a hyperbola with transverse axis on the line $v = 0$, and conjugate axis on the line $u = 0$. The length of the transverse axis is approximately $2(1.76) = 3.52$ and the length of the conjugate axis is approximately $2(1.38) = 2.76$. Thus, the vertices are $(u,v) \approx (\pm 1.76, 0)$, and the foci are $(u,v) \approx (\pm 2.23, 0)$. In xy-coordinates, the transverse axis is on the line $y = \tan(\pi/8)x$ (since $\theta = \pi/8$) and the conjugate axis is on the line $y = -\cot(\pi/8)x$. Using formulas (4), the xy-coordinates of the vertices are approximately $\pm(1.62, 0.67)$, and the foci are approximately $\pm(2.06, 0.855)$. See Figure 4.

For an interesting exercise, use the techniques and formulas given in this section to prove the following result:

Theorem 1. The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

(a) defines a parabola if $B^2 - 4AC = 0$.
(b) defines an ellipse if $B^2 - 4AC < 0$.
(c) defines a hyperbola if $B^2 - 4AC > 0$.

(Hint: First show that for any θ, $B^2 - 4AC = b^2 - 4ac$.)

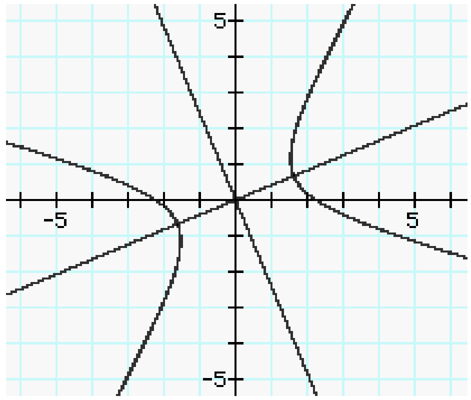

Figure 4

Exercises

In problems 1-4, identify the conic section and find a suitable first quadrant angle so that the resulting rotation of axes eliminates the mixed term.

1. $3x^2 - 2xy + y^2 + 4x + 2y - 1 = 0$
2. $4x^2 + 3\sqrt{3}xy + y^2 = 3$
3. $x^2 + 2xy + y^2 + 8x - 8y = 0$
4. $x^2 + \sqrt{3}xy + 4y^2 + 5\sqrt{5}y + 5 = 0$

In problems 5-10, find a suitable first quadrant angle so that the resulting rotation of axes eliminates the mixed term. Find the new equation and identify the conic section. If it is a parabola, find the axis, vertex, focus, and directrix. If it is an ellipse, find the lines containing the major and minor axes, lengths of major and minor axes, center, vertices, and foci. If it is a hyperbola, find the lines containing the transverse and conjugate axes, lengths of transverse and conjugate axes, center, vertices, and foci. The problems marked * are more difficult.

5. $xy = 1$
6. $13x^2 - 6\sqrt{3}xy + 7y^2 = 16$
7. $x^2 - 2\sqrt{3}xy + 3y^2 - \sqrt{3}x - y = 0$
8. $11x^2 + 10\sqrt{3}xy + y^2 = 32$

*9. $4x^2 - 2xy + 4y^2 - 6\sqrt{2}x - 6\sqrt{2}y = 3$

*10. $4x^2 - 4xy + y^2 + 5\sqrt{5}x + 5 = 0$

Answers to the Exercises

1. Theorem 1 implies that the conic is an ellipse since $B^2 - 4AC = -8 < 0$.
$\cot 2\theta = -1$ implies that $\theta = 3\pi/8$.

2. Theorem 1 implies that the conic is a hyperbola since $B^2 - 4AC = 11 > 0$.
$\cot 2\theta = \frac{1}{\sqrt{3}}$ implies that $\theta = \pi/6$.

3. Theorem 1 implies that the conic is a parabola since $B^2 - 4AC = 0$.
$\cot 2\theta = 0$ implies that $\theta = \pi/4$.

4. Theorem 1 implies that the conic is an ellipse since $B^2 - 4AC = -13 < 0$.
$\cot 2\theta = -\sqrt{3}$ implies that $\theta = 5\pi/12$.

5. $\theta = \pi/4$, and the uv-equation is $\frac{u^2}{2} - \frac{v^2}{2} = 1$. This is a hyperbola with center $(0, 0)$, transverse axis on the line $y = x$, conjugate axis on the line $y = -x$, vertices $(x, y) = \pm(1, 1)$, and foci $(x, y) = \pm(\sqrt{2}, \sqrt{2})$. The length of the transverse axis is $2\sqrt{2}$ and the length of the conjugate axis is $2\sqrt{2}$.

6. $\theta = \pi/3$, and the uv-equation is $\frac{u^2}{4} + \frac{v^2}{1} = 1$. This is an ellipse with center $(0, 0)$, major axis on the line $y = \sqrt{3}x$, minor axis on the line $y = -x/\sqrt{3}$, vertices $(x, y) = \pm(1, \sqrt{3})$, and foci $(x, y) = \pm(\sqrt{3}/2, 3/2)$. The length of the major axis is 4 and the length of the minor axis is 2.

7. $\theta = \pi/6$, and the uv-equation is $v^2 = \frac{u}{2}$. This is a parabola with vertex $(x, y) = (0, 0)$, axis $y = x/\sqrt{3}$, focus $(x, y) \approx (.1083, .0625)$, and directrix $y = -\sqrt{3}x - .25$.

8. $\theta = \pi/6$, and the uv-equation is $\frac{u^2}{2} - \frac{v^2}{8} = 1$. This is a hyperbola with center $(0, 0)$, transverse axis on the line $y = x/\sqrt{3}$, conjugate axis on the line $y = -\sqrt{3}x$, vertices $(x, y) = \pm(\sqrt{6}/2, \sqrt{2}/2)$, and foci $(x, y) = \pm(\sqrt{30}/2, \sqrt{10}/2)$. The length of the transverse axis is $2\sqrt{2}$ and the length of the conjugate axis is $4\sqrt{2}$.

9. $\theta = \pi/4$, and the uv-equation is $\frac{(u-2)^2}{5} + \frac{v^2}{3} = 1$. This is an ellipse with center $(\sqrt{2}, \sqrt{2})$, major axis on the line $y = x$, minor axis on the line $y = -x + 2\sqrt{2}$, vertices $(x, y) = (\frac{2+\sqrt{5}}{\sqrt{2}}, \frac{2+\sqrt{5}}{\sqrt{2}})$ and $(\frac{2-\sqrt{5}}{\sqrt{2}}, \frac{2-\sqrt{5}}{\sqrt{2}})$, and foci $(x, y) = (\frac{2+\sqrt{2}}{\sqrt{2}}, \frac{2+\sqrt{2}}{\sqrt{2}})$ and $(\frac{2-\sqrt{2}}{\sqrt{2}}, \frac{2-\sqrt{2}}{\sqrt{2}})$. The length of the major axis is $2\sqrt{5}$ and the length of the minor axis is $2\sqrt{3}$.

10. $\cos 2\theta = -3/5$ and $\sin 2\theta = 4/5$, so $\theta = \frac{1}{2}\arccos(-\frac{3}{5}) \approx 63.44°$. By the half-angle formulas, $\cos\theta = \sqrt{5}/5$ and $\sin\theta = 2\sqrt{5}/5$. Substituting into formulas (4) yields the uv-equation $(v-1)^2 = -u$. This is a parabola with approximate axis $y = 2x + 2.235$ (note that $\tan\theta = 2$), vertex $(x, y) \approx (-.894, .447)$, focus $(x, y) \approx (-1.006, .224)$, and approximate directrix $y = -0.5x + 1.062$.