LECTURE 2 Second order Systems (Updated 1/15/2019)

In this set of notes we address the properties of the second order, constant coefficient differential equation:

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{f}(t) + b_o f(t) \quad ; y(0) = y_o \quad \& \quad \dot{y}(0) = v_o . \tag{1}$$

This equation describes the relation between an *input f*(*t*) and an *output y*(*t*). We will restrict our attention to the solution of (1) via Laplace transforms. Furthermore, we will assume here that all initial conditions are zero. The reader will address the case in which they are not in the homework problems. Taking the Laplace transform of (1), we obtain: $(a_2 s^2 + a_1 s + a_0)Y(s) = (b_1 s + b_o)F(s)$. It follows that the system *transfer function* is:

$$H(s) \overset{\Delta}{=} \frac{Y(s)}{F(s)} = \frac{b_1 s + b_o}{a_2 s^2 + a_1 s + a_0} . \tag{2}$$

For any *input* $f(t)$ the solution of (1) under zero initial conditions is, in the *s*-domain, $Y(s) = H(s)F(s)$.

***The Unit Impulse Response of a Second Order System*:** For $f(t) = \delta(t)$ we have $Y(s) = H(s)$. Notice that the units of (2) is the ratio of the output to the input. Hence, while $Y(s) = H(s)$ is mathematically correct, one must note that the units of $Y(s) = H(s)$ as the solution for a unit impulse are those of the output. This relation is of such fundamental importance that we highlight it as

An Important Result- The system impulse response and transfer function constitute a Laplace transform pair. Even so, the transfer function units are those of the output divided by those of the input, while the impulse response units are those of the output alone.

The system (2) is a *stable* system if the impulse response $h(t) \to 0$ as $t \to \infty$. This requires that the two roots of the *characteristic polynomial* $A(s) = a_2 s^2 + a_1 s + a_0$ both be in the *Left Half Plane* (*LHP*). A first order system has only one root, and it must be a real number. In contrast, a second order system can have either (i) two real roots, or (ii) a pair of complex-conjugate roots. We now consider these two cases. Furthermore, we will assume that the system is stable.

***Case* 1 The roots of $A(s) = a_2 s^2 + a_1 s + a_0$ are real:** Here, we will assume that the two real roots are not equal. The reader will address the case where they are equal in the homework problems. The roots of $A(s) = a_2 s^2 + a_1 s + a_0$ are the values of *s* that make it zero. We will express this as:

$$0 = a_2 s^2 + a_1 s + a_0 = s^2 + (a_1 / a_2)s + (a_0 / a_2) \overset{\Delta}{=} (s + \omega_1)(s + \omega_2) . \tag{3}$$

The roots are $s_1 = -\omega_1$ and $s_2 = -\omega_2$. Stability requires that both roots be real and negative, and so the parameters ω_1 and ω_2 are positive real numbers. Furthermore, they are the inverses of the system *time constants*, $\tau_1 = 1/\omega_1$ and $\tau_2 = 1/\omega_2$. In this case, the system transfer function (2) becomes:

$$H(s) = \frac{1}{a_2} \bullet \left(\frac{b_1 s + b_o}{(s + \omega_1)(s + \omega_2)} \right) = \frac{1}{a_2} \bullet \left(\frac{b_1 s}{(s + \omega_1)(s + \omega_2)} + \frac{b_o}{(s + \omega_1)(s + \omega_2)} \right) . \tag{4}$$

To obtain the system impulse response associated with (4), we need the following Laplace table entries:

#13: $\dfrac{b-a}{(s+a)(s+b)} \leftrightarrow e^{-at} - e^{-bt}$ and #16: $\dfrac{s(b-a)}{(s+a)(s+b)} \leftrightarrow be^{-bt} - ae^{-at}$.

Note that since multiplication by s in the Laplace domain is equivalent to taking the derivative in the time domain. Hence, #16 should be readily obvious from #13. We now have the denominator of $H(s)$ in the appropriate form for both #13 and #16, but we have a ways to go. Using #13, the far right term in (4) is :

$$\frac{b_o}{(s+\omega_1)(s+\omega_2)} = \left(\frac{b_0}{\omega_2-\omega_1}\right)\bullet\frac{\omega_2-\omega_1}{(s+\omega_1)(s+\omega_2)} \quad\leftrightarrow\quad \left(\frac{b_0}{\omega_2-\omega_1}\right)\bullet(e^{-\omega_1 t}-e^{-\omega_2 t})\,.$$

Using #16, the other term is:

$$\frac{b_1 s}{(s+\omega_1)(s+\omega_2)} = \left(\frac{b_1}{\omega_2-\omega_1}\right)\bullet\frac{(\omega_2-\omega_1)s}{(s+\omega_1)(s+\omega_2)} \quad\leftrightarrow\quad \left(\frac{-b_1}{\omega_2-\omega_1}\right)\bullet(\omega_1 e^{-\omega_1 t}-\omega_2 e^{-\omega_2 t})\,.$$

Hence, $$h(t)=\frac{1}{a_2(\omega_2-\omega_1)}\left[b_0(e^{-\omega_1 t}-e^{-\omega_2 t})-b_1(\omega_1 e^{-\omega_1 t}-\omega_2 e^{-\omega_2 t})\right].$$

This can be rearranged as: $$h(t)=\frac{1}{a_2(\omega_2-\omega_1)}\left[(b_0-b_1\omega_1)e^{-\omega_1 t}+(b_1\omega_2-b_0)e^{-\omega_2 t}\right]. \qquad (5)$$

And so, we see that the system impulse response is the sum of two decaying exponential functions. The relative contribution of each one will be determined both by which one decays slower and by the relative weights. Often, it is presumed that the slower one will dominate the shape of (5). However, if the faster one has a much larger weight, then this will not be the case. The nice thing about having (5) is that it shows you the exact nature of the weights: $$q_1 \triangleq (b_0-b_1\omega_1) \quad and \quad q_1 \triangleq (b_1\omega_2-b_0)\,. \qquad (6)$$

Conclusion: A second order system whose characteristic polynomial as negative real roots has an impulse response that is the sum of two decaying exponentials whose relative weights are given by (6).

Definition 1. A second order stable system having two real roots is called an ***overdamped*** system. If the roots are real and equal, it is called a ***critically damped*** system. If the roots are a complex-conjugate pair, then it is called an ***underdamped*** system.

***<u>Case 2</u>* The roots of $A(s)=a_2s^2+a_1s+a_0$ are a complex conjugate pair:** Recall from the 'assumed solution' technique that for a second order system, the complementary (not complimentary! 

) solution has the form $y_c(t)=C_1e^{s_1t}+C_2e^{s_2t}$. Suppose that $s_1=\sigma+i\omega$, and that $s_2=\sigma-i\omega\triangleq\bar{s}_1$. Since $y_c(t)$ is a real-valued function of time, we must then have $C_2=\bar{C}_1$. Write $C_1=|C|e^{i\phi}$. Then: $y_c(t)=|C|e^{\sigma t}(e^{i(\omega t+\phi}+e^{-i(\omega t+\phi)})=2|C|e^{\sigma t}\cos(\omega t+\phi)$. In words, this solution will oscillate at frequency ω, and decay exponentially if $\sigma<0$ [i.e. if the system poles are in the proper Left Half Plane (LHP)].

***A 'mantra' for* AERE331:** The characteristic polynomial for an underdamped second order system can always be written as $p(s)=s^2+2\zeta\omega_n s+\omega_n^2$.

***Example* 1.** Consider the system described by the following differential equation:

$$\ddot{y}+0.1\dot{y}+25y=f(t) \quad ; y(0)=y_o \;\; \dot{y}(0)=v_o$$

(a) Give the system transfer function.

<u>*Answer*</u>: $\dfrac{Y(s)}{F(s)}\triangleq G_p(s)=\dfrac{1}{s^2+0.1s+25}$

(b) Without actually computing the system poles [i.e. the roots of the polynomial $p(s) = s^2 + 0.1s + 25$] determine whether the system is overdamped or underdamped:
Answer: Recall that, from the *quadratic formula*, the roots will be complex, if "$b^2 < 4ac$". In relation to *p*(*s*) we have $0.1^2 = 0.01 \quad << \quad 4(1)(25) = 100$. Hence, the poles are complex, and hence the system is underdamped.

(c) You should have found that the system poles are complex. Rather than using the quadratic formula to compute them, recall (see the above 'mantra') that for a second order underdamped system, $p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$. Use the method of equating coefficients to find the undamped natural frequency, ω_n, and then the damping ratio, ζ
Solution: $p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 0.2s + 25 \;\Rightarrow\; \omega_n = 5 \;\Rightarrow\; 2\zeta\omega_n = 2\zeta(5) = 0.1 \;\Rightarrow\; \zeta = 0.01.$

(d) Compute the system damped natural frequency, time constant, settling time, percent overshoot, rise time, and static gain. [*For the following quantities, see the discussion in the book*.]
Solution: $\omega_d \overset{\Delta}{=} \omega_n\sqrt{1-\zeta^2} = 5\sqrt{1-10^{-4}} \cong 5$ rad/sec ; $\tau \overset{\Delta}{=} \dfrac{1}{\zeta\omega_n} = \dfrac{1}{0.01(5)} = 20$ sec.

The 4-τ settling time is $t_s = 80$ sec ; $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \cong e^{-\pi\zeta} = e^{0.01\pi} \cong 0.97$ or 97%.

The rise time is $t_r \cong \dfrac{1.8}{\omega_n} = \dfrac{1.8}{5} \cong 0.36$ sec. The static gain is $G_p(s=0) = 1/25 = 0.04$

(e) Use the unit step response plot to verify the information in part (d).
Solution: To obtain the step response using Matlab, we first define the system transfer function:

```
>> Gp=tf(1,[1 .1 25])
 >>Transfer function:
      1
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s^2 + 0.1 s + 25
```

To get the unit step response for this system, we type
>> step(Gp)

You should be able to use the data cursor to verify the information in (d).

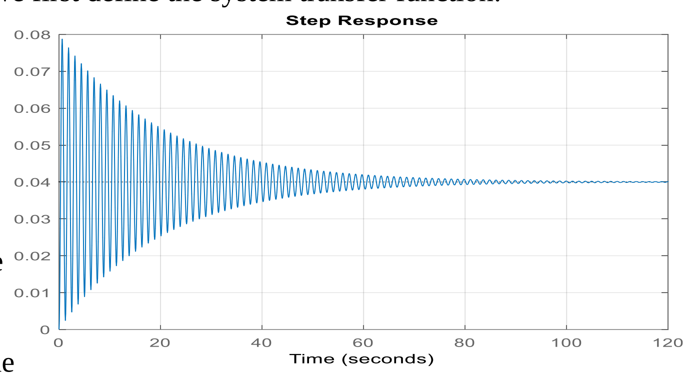

Figure 2.1 Plant unit step response.

(f) Plot the plant FRF using the Matlab command 'bode(sys).
Solution: $G_p(i\omega) = \dfrac{1}{(i\omega)^2 + 0.1(i\omega) + 25} = \dfrac{1}{(25-\omega^2) + i(0.1\omega)}$.
Let $G_p(i\omega) = M(\omega)e^{i\theta(\omega)}$ where the magnitude

$$M(\omega) = \sqrt{\frac{1}{(25-\omega^2)^2 + (0.1\omega)^2}} \quad \text{with} \quad M(\omega)_{dB} = 20\log_{10}[M(\omega)]$$

and the phase $\theta(\omega) = \begin{cases} -a\tan[0.1\omega/(25-\omega^2)] \;\; for\; \omega < 5 \\ \pi - a\tan[0.1\omega/(\omega^2 - 25)] \;\; for\; \omega > 5 \end{cases}$.

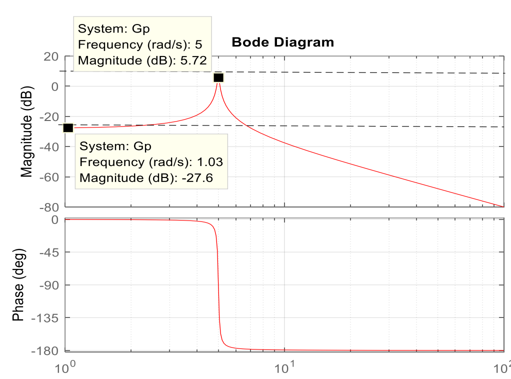

Figure 2.2 Plant FRF.

(g) The plant low frequency gain is ~ -27.6dB. The amplification at resonance is approximately 5.72 – (-27.6) =33.32 dB (i.e. M=46.345). Hence, for a sinusoidal input with unity amplitude and frequency 5 rad/sec, the plant steady state response will be a sinusoid at the same frequency, but with amplification (re: low frequency, or static gain) of almost 50!!! [i.e. the plant will resonate]. From the above FRF, we see that the response level at resonance is about 5.72dB or [redacted] 1.93. Use the Matlab 'lsim' command to verify that, for an input $f(t) = \sin(5t)$ the steady state output will be $y(t) = 1.93\sin(5t - \pi/2)$.

<u>*Solution*</u>: The commands are: >> t=0:.01:150; >> f=sin(5*t); >> y=lsim(Gp,f,t); >> plot(t,f) >> hold on >> plot(t,y,'r')

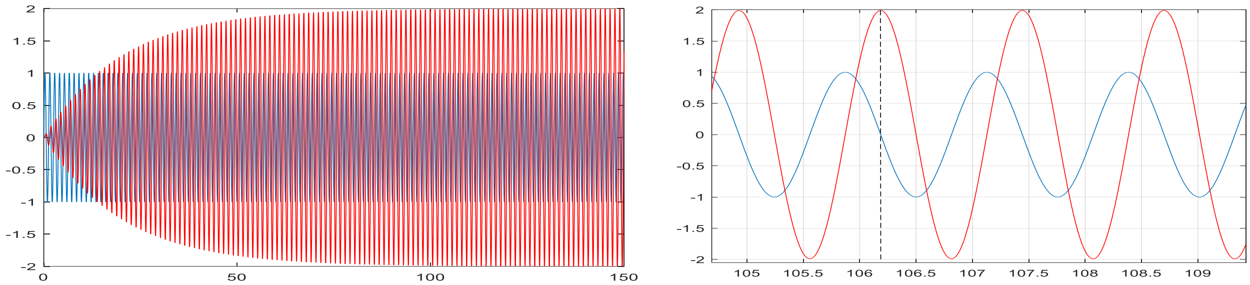

Figure 2.3 Sinusoidal input (blue) and response (red).

<u>Example 2</u> **[This and the following example are from Sherman's AERE355 Ch.4 notes]**

It is desired to develop a test rig for estimating the pitching lift derivative coefficient, C_{L_α}, for horizontal tail designs. The beginning of this development is shown below.

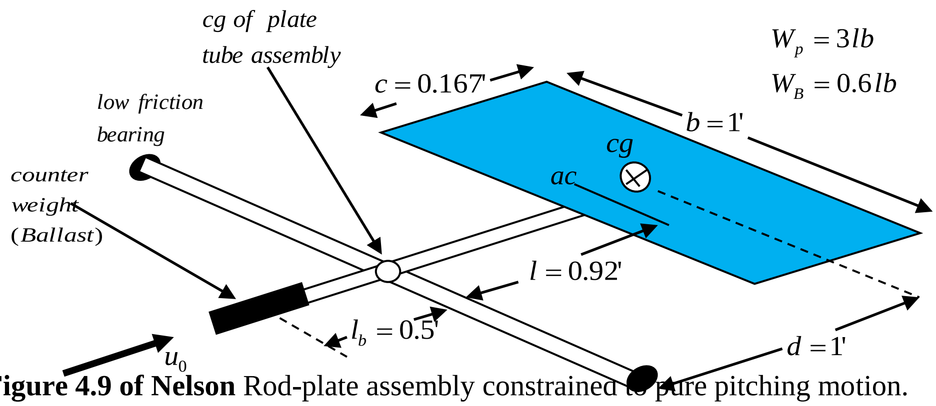

Figure 4.9 of Nelson Rod-plate assembly constrained to pure pitching motion.

(a) ***Development of the equation of angular motion*:**

We begin with: $$\sum M_{cg} = I_y \ddot{\theta} = \left(\frac{\partial M}{\partial \alpha}\right)_0 \alpha + \left(\frac{\partial M}{\partial q}\right)_0 q \quad \text{(E1.1)}$$

Note #1: Since the *cg* of the test rig is constrained, the angle of attack, α, and the pitch angle, θ, are one and the same. Hence, $\alpha = \theta$ and $q = \dot{\theta}$.

Note #2: Let $M_\alpha \stackrel{\Delta}{=} \left(\frac{\partial M}{\partial \alpha}\right)_0 \times \frac{1}{I_y}$ and $M_q \stackrel{\Delta}{=} \left(\frac{\partial M}{\partial q}\right)_0 \times \frac{1}{I_y}$.

From these notes, (E1.1) can be written as:

$$\ddot{\theta} - M_q \dot{\theta} - M_\alpha \theta = 0. \quad \text{(E1.2)}$$

(b) ***Development of the relation between*** (M_α, M_q) ***and*** C_{L_α}:

$$M(\alpha) = -l \times (C_{L_\alpha}\alpha) \times \left(\frac{1}{2}\rho u_0^2 S\right) \quad \Rightarrow \quad M_\alpha \overset{\Delta}{=} \frac{dM(\alpha)}{d\alpha} = -C_{L_\alpha}\left(\frac{1}{2}\rho u_0^2 Sl\right)/I_y . \tag{E1.3a}$$

Recall that $\tan\alpha \cong ql/u_0$, so that for small α, we have $\alpha \cong ql/u_0$. Hence,

$$M(q) = -l \times \left(C_{L_\alpha} \times \frac{ql}{u_0}\right) \times \left(\frac{1}{2}\rho u_0^2 S\right) \quad \Rightarrow \quad M_q \overset{\Delta}{=} \frac{dM(q)}{dq} = -C_{L_\alpha}\left(\frac{l}{u_0}\right)\left(\frac{1}{2}\rho u_0^2 Sl\right)/I_y . \tag{E1.3b}$$

From (E1.3a-b) we obtain:

$$M_q = \left(\frac{l}{u_0}\right) M_\alpha . \tag{E1.3c}$$

Substituting (E1.3c) into (E1.2) gives:

$$\ddot{\theta} - \left(\frac{l}{u_0}\right) M_\alpha \dot{\theta} - M_\alpha \theta = 0 . \tag{E1.4}$$

(c) *Relation between M$_\alpha$ and the transient response associated with* (E1.4):

[Subtitled: *How many different way can we estimate C_{L_α} ?*]

We know that the transient response associated with (4) will be decaying and oscillatory. Hence, we can write the left side of (E1.4) as:

$$\ddot{\theta} + \left(\frac{l}{u_0}\right) M_\alpha \dot{\theta} + M_\alpha \theta \quad \overset{\Delta}{=} \quad \ddot{\theta} + 2\varsigma\omega_n\dot{\theta} + \omega_n^2\theta . \tag{E1.5a}$$

From (E1.5a), we have:

$$\omega_n = \sqrt{M_\alpha} . \tag{E1.5b}$$

We also have: $\left(\frac{l}{u_0}\right) M_\alpha = \left(\frac{l}{u_0}\right)\omega_n^2 = 2\varsigma\omega_n$. This gives: $\quad \varsigma = \omega_n l / 2u_0 = \sqrt{M_\alpha}/2u_0$. (E1.5c)

Also, the system *time constant* is: $\tau = \dfrac{1}{\varsigma\omega_n} = \dfrac{2u_0}{M_\alpha}$.

Now, the transient response associated with (E1.5a) has the form:

$$\theta(t) = \theta_1\, e^{-\varsigma\omega_n t} \sin(\omega_d t + \varphi_1) \tag{E1.6a}$$

where the pair of constants (θ_1, φ_1) will depend on the type of specified initial conditions . The initial conditions are not as important as the nature of the response (E1.6a). If we define $\tau \overset{\Delta}{=} 1/\varsigma\omega_n$, then (E1.6a) becomes:

$$\theta(t) = \theta_1\, e^{-t/\tau} \sin(\omega_d t + \varphi_1) . \tag{E1.6b}$$

The parameter τ is called the *time constant* associated with (E1.5a). This gives rise to one method for estimating C_{L_α} from the transient response measurement.

***Method 1**. Ignore the oscillations and use only the decay envelope* $\theta_1\, e^{-t/\tau}$: At a time $t = \tau$, this envelope will equal $\theta_1\, e^{-1} \cong 0.37\theta_1$. And so, to estimate $t = \tau$, we sketch an exponential envelope on the decaying oscillatory response, and find the time at which this envelope is ~37% of its peak value. Having this estimate, call it $\hat{\tau}$, we then note that

$$\frac{1}{\tau} = \left(\frac{l}{2u_0}\right) M_\alpha = C_{L_\alpha}\left(\frac{\rho u_0 S l^2}{4I_y}\right).$$

And so, our estimate of the magnitude of C_{L_α}, call it $\hat{C}_{L_\alpha}$ is: $\quad \hat{C}_{L_\alpha} = \left(\dfrac{\rho u_0 S l^2}{4I_y}\right)^{-1} \dfrac{1}{\hat{\tau}}$.

Method 2. *Ignore the envelope and use only the oscillation frequency*: The oscillation frequency, ω_d, of the decaying response (E1.6) is: $\omega_d = \omega_n\sqrt{1-\varsigma^2}$. From (E1.5b) and (E1.5c) this becomes:

$$\omega_d = \sqrt{M_\alpha}\sqrt{1-M_\alpha/(2u_0)^2} \,.$$

And so, for an estimate of ω_d, call it $\hat{\omega}_d$, this equation provides an estimate of M_α, call it $\hat{M}_\alpha$. That estimate and (E1.3a) results in the desired estimate of C_{L_α}, call it $\hat{C}_{L_\alpha}$.

Method 3. *Use both decay and oscillation frequency information*: This method is best described using a plot of (E1.6): [This is called the *modified log-decrement method.*]

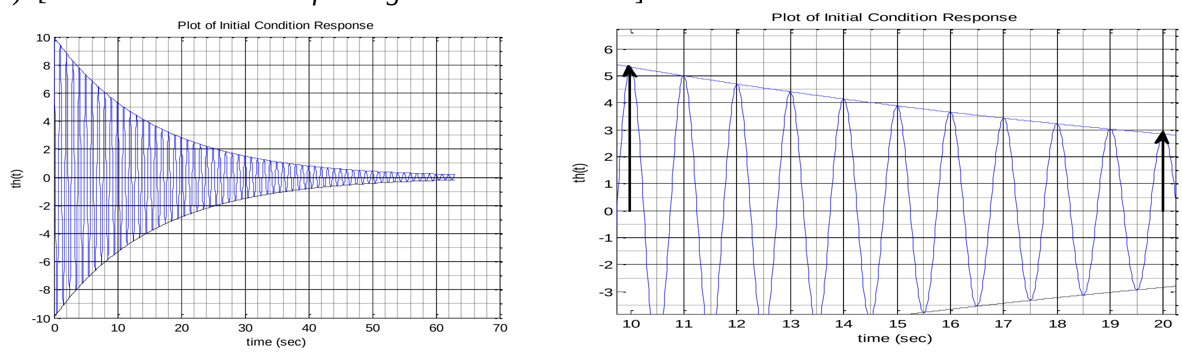

Figure 1. Initial condition response for $\omega_n = 2\pi$ and $\varsigma = 0.01$.

The plot on the right in figure 1 shows that for $t_0 = 10\,\text{sec}$, $\theta(t_0) \cong 5.4^o$, and for $t_1 = 20\,\text{sec}$, $\theta(t_1) \cong 3^o$. The duration, $t_1 - t_0$ between these times is 20 sec., but it is also 10 *cycles* (or *periods*), where one period is $T = 2\pi/\omega_d = 2\pi/\omega_n\sqrt{1-\varsigma^2}$. Hence, $t_1 - t_0 = 20\pi/\omega_n\sqrt{1-\varsigma^2}$.

Now, from (6a), we have $\theta(t_0) \cong \theta_0\, e^{-\varsigma\omega_n t_0}$ and $\theta(t_1) \cong \theta_1\, e^{-\varsigma\omega_n t_1}$. And so:

$$\frac{\theta(t_0)}{\theta(t_1)} = \frac{e^{-\varsigma\omega_n t_0}}{e^{-\varsigma\omega_n t_1}} = e^{-\varsigma\omega_n(t_1-t_0)} = e^{20\pi\varsigma/\sqrt{1-\varsigma^2}} \,.$$

Define $\delta \overset{\Delta}{=} \varsigma/\sqrt{1-\varsigma^2}$. Then we have $\delta = \ln\left(\dfrac{\theta(t_0)}{\theta(t_1)}\right)/20\pi = \ln(5.4/3)/20\pi = 0.0094$. From the definition of δ, we have $\varsigma = \delta/\sqrt{1-\delta^2}$. And so, our estimate $\hat{\delta} = 0.0094$ gives: $\hat{\varsigma} \cong 0.0094$. This estimate is close to the true value $\varsigma = 0.01$, and it would have been closer, had we not used visually-based estimates of $\theta(t_0)$ and $\theta(t_1)$.

Finally, we can use (E1.3a), (E1.5b) and (E1.5c) to obtain: $\hat{C}_{L_\alpha} = \left(\dfrac{8I_y}{\rho Sl}\right)\hat{\varsigma}^2$.

Notice that in this method it is not necessary to measure u_0, since the above estimate does not involve it. This can offer a significant advantage over other methods, both in terms of accuracy and equipment.

(d) ***Use of the setup in Figure 4.9 to validate the experimental design*****:** Now that we know how to use the transient response, (6) to estimate the lift coefficient derivative for a given tail design, it is necessary to validate the experimental setup illustrated in Figure 4.9. In that design, we are assuming that the cross-bar bearing friction is negligible. We will also assume that the tube that supports the tail is completely rigid, and that its aerodynamic influence is negligible. Finally, we will assume that we have perfect measurements of the geometric and mass quantities described in that figure. By using a flat plate with known lift properties, we can compute the theoretical value for C_{L_α}. If the experimentally measured transient response matches our theoretical prediction reasonably well, then we can assume that we have a valid setup. We will now proceed to compute the theoretical value for C_{L_α} from the given numerical information.

***The moment of inertia, I_y*:** From the *parallel axis theorem*, we have: $I_y = I_{y'} + m_p l^2$. The moment of inertia of the flat plate about its *cg*, and the moment of the plate about the setup *cg* are:
$I_{y'} = (1/12)\rho btc^3 = (1/12)(W_p/g)c^2 = 2.16\times 10^{-5} slug-ft^2$ & $m_p l^2 = (W_p/g)l^2 = 9.3\times 10^{-3} slug-ft^2$.
Hence, $I_y^{(plate)} = 9.32\times 10^{-3} slug-ft^2$. We also have $I_y^{(Ballast)} = 4.60\times 10^{-3} slug-ft^2$. Hence, the moment of inertia of the entire system is: $I_y \cong 1.4\times 10^{-2} slug-ft^2$.

***Numerical values for M_α and M_q*:**
For an infinite flat plate, we have $C_{L_\alpha}^{(\infty)} = 2\pi/rad$. For a finite plate with aspect ratio, *AR*, the lift coefficient derivative is: $C_{L_\alpha} = C_{L_\alpha}^{(\infty)}/[1 + C_{L_\alpha}^{(\infty)}/(\pi\ AR)]$. Since $AR = 6$, we have $C_{L_\alpha} = 4.7/rad$
Using these highlighted results and equations (3), we obtain: $M_\alpha = -36.1/s^2$ and $M_q = -1.38/s^2$

Hence, (2) becomes: $\ddot{\theta} + 1.38\dot{\theta} + 36.1\theta = 0$. It follows the theoretical values for ω_n and ς are: $\omega_n = 6.0\ rad/\sec$ and $\varsigma = 0.038$. If we give this plate an initial angular displacement, $\theta_1 = 10^0$, then we should expect the experimentally measured transient response to be similar to the plot below.

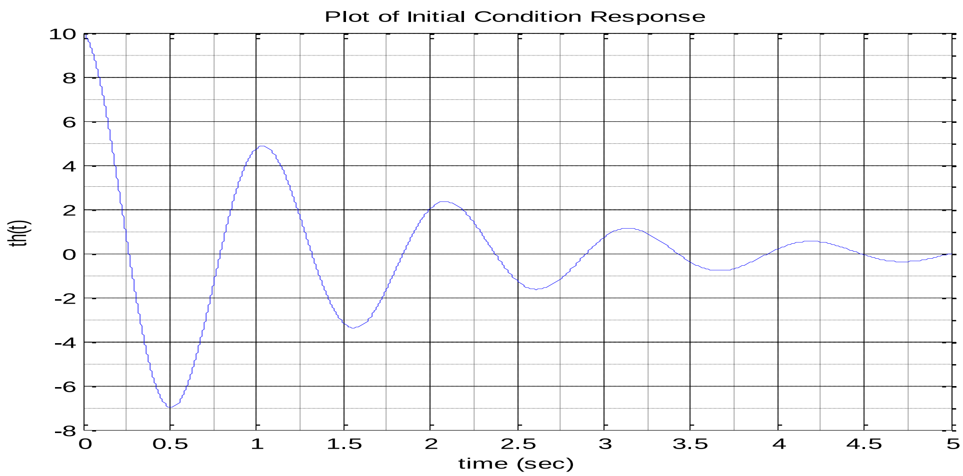

Figure 2. Expected description of the experimentally measured transient response of the flat plat to an initial angular displacement of 10°. □

***Example* 3 [Related to Nelson EXAMPLE PROBLEM 5.2 on p.190]** For a plane constrained to pure yawing, we have:

$$\Delta\ddot{\psi} - N_r\Delta\dot{\psi} + N_\beta\Delta\psi = N_{\delta_r}\Delta\delta_r. \qquad (5.23).$$

Write this as:

$$\Delta\ddot{\psi} + 2\zeta\omega_n\Delta\dot{\psi} + \omega_n^2\Delta\psi = N_{\delta_r}\Delta\delta_r$$

We are given: $N_\beta = C_{n_\beta}(QSb/I_z) = \omega_n^2$ and $N_r = C_{n_r}[(b/2u_0)QSb/I_z] = 2\zeta\omega_n$. Hence,

$$\omega_n = \sqrt{N_\beta}\text{, and } \zeta = -0.5N_r/\sqrt{N_\beta}\text{. [See bottom of p.192.]}$$

What the author does not do, we will do now; namely to understand the specific parameters that control the dynamics. Recall that $Q = 0.5\rho u_0^2$. Hence:

$$\omega_n = u_0\sqrt{0.5C_{n_\beta}\rho Sb/I_z)} \quad , \quad \zeta = -0.5\frac{C_{n_r}}{\sqrt{C_{n_\beta}}}\sqrt{\rho Sb/I_z} \text{ and } \tau = \frac{1}{\zeta\omega_n} = \frac{2}{N_r} = \frac{8I_z}{u_0(C_{n_r}Sb^2\rho)}$$

So, as the plane speed increases, ω_n increases, τ decreases, and ζ is not affected.

QUESTION: Why is this information valuable to a pilot?

ANSWER: One reason is obvious. At higher speeds the plane lateral dynamics will be faster (i.e. τ is small). Hence, it will be more difficult for the pilot (who has relatively slow dynamics) to accommodate the plane dynamics. If this happens, the pilot could reduce his speed to better accommodate these dynamics. Another reason is that planes often fly through spatial turbulence. The frequencies of the temporal turbulence are directly proportional to u_0. Suppose that the wing has a low-damped structural resonance at a frequency $\omega_{n_{wing}}$. If at a given speed, the frequency $\omega_{n_{wing}}$ is close to the frequency ω_n, a cross-coupling of these two resonances could occur; making for a very difficult situation.

Now: $C_{n_\beta} = C_{n_{\beta_{wf}}} + \eta_v V_v (1 + d\sigma / d\beta) C_{L_{\alpha_v}}$ and $C_{n_r} = -2\eta_v V_v (l_v / b) C_{L_{\alpha_v}}$, where $V_v = S_v l_v / Sb$.
Clearly, there are many variables, in addition to u_0, that affect the lateral dynamics. For example, what would happen if a portion of the vertical tail were to break off? This would reduce the values of $C_{L_{\alpha_v}}$, V_v (via reduction of both S_v and l_v), and l_v (directly). Both C_{n_β} and C_{n_r} would be reduced in magnitude. Consequently, ω_n would decrease and τ would increase. What would happen to ζ is a more complicated question.

In summary, the question of just how the various parameters influence the flight dynamics is a rich and complicated one. But it is a question that is well-posed in the context of the subject of flight dynamics and control. □

[NOTE: I will not cover the specific questions related to EXAMPLE PROBLEM 5.2 in class. They are included below. Feel free to go through it on your own. I included it to give students interested in flight dynamics further insight into the topic. We may well return to it when we address state space models later in this course.] Suppose an airplane is constrained to a pure yawing motion. Use the data for the general aviation airplane in Appendix B, determine the following quantities:
(a) The yaw moment equation written in state space form.
(b) The characteristic equation and eigenvalues for the system.
(c) The damping ratio, ς, and undamped natural frequency, ω_n.
(d) The response of the plane to a 5° rudder input. Assume initial conditions are: $\Delta\beta(0) = \Delta r(0) = 0$.

Solution: The solution that follows is taken directly from pp.189-191 of Nelson: From p.189 we have

$$\Delta\ddot{\psi} + (N_r - N_{\dot{\beta}})\Delta\dot{\psi} + N_\beta \Delta\psi = N_{\delta_r}\Delta\delta_r . \qquad (5.23)$$

For a sea level flight condition, the weathercock stability coefficient, the yaw damping coefficient, and the rudder control power coefficient have, respectively, the following values:

$$C_{n_\beta} = 0.071 / r \quad ; \quad C_{n_r} = -0.125 / r \quad ; \quad C_{n_{\delta_r}} = -0.072 / r .$$

[The derivative $C_{n_{\dot{\beta}}}$ is not included in the appendix, and will be assumed to be zero.]

For $u_0 = 176\ ft/s$, the dynamic pressure at sea level is: $Q \overset{\Delta}{=} 0.5\rho u_0^2 = 36.8\ lb/ft^2$.
The plane geometry parameters include: $S = 184\ ft^2 \quad ; \quad b = 33.4\ ft \quad ; \quad I_z = 3530\ slug - ft^2$.
These values result in the following dimensional derivative values:

$N_\beta = C_{n_\beta}(QSb/I_z) = 4.55/s^2$; $N_r = C_{n_r}[(b/2u_0)QSb/I_z] = -0.76/s$; $N_{\delta_r} = C_{n_{\delta_r}}(QSb/I_z) = -4.6/s^2$

Substituting these into (5.23) gives:

$$\Delta\ddot{\psi} \quad + \quad 0.76\Delta\dot{\psi} \quad + \quad 4.55\Delta\psi \quad = \quad -4.6\Delta\delta_r \, . \qquad (1)$$

We are now in a position to solve parts (a-d).

(a) The yaw moment equation written in state space form.

Solution: Recall that the state space form is: $\dot{x} = Ax + Bu$. In relation to (1) above, this will be a 2-D system. Hence, $x(t) = [x_1(t)\ x_2(t)]^{tr}$ We know that one state, say, $x_1(t)$, will be $\Delta\psi(t)$. We also know that, since the plane is constrained to pure yawing, $\Delta\dot{\psi}(t) = \Delta r(t)$. And so, let $x_2(t) = \Delta r(t)$. From (1) we have: $\Delta\dot{r} \quad + \quad 0.76\Delta r \quad + \quad 4.55\Delta\psi \quad = \quad -4.6\Delta\delta_r$. And so, we arrive at:

$$\begin{bmatrix} \Delta\dot{r} \\ \Delta\dot{\psi} \end{bmatrix} = \begin{bmatrix} -0.76 & -4.55 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta\psi \end{bmatrix} + \begin{bmatrix} -4.6 \\ 0 \end{bmatrix} \Delta\delta_r \ .$$

(b) The characteristic equation and eigenvalues for the system.

Solution: The characteristic polynomial is simply: $p(s) = s^2 \quad + \quad 0.76s \quad + \quad 4.55$. The eigenvalues of $A = \begin{bmatrix} -0.76 & -4.55 \\ 1 & 0 \end{bmatrix}$ are obtained via *Matlab*:

```
> A=[-.76 -4.55 ; 1 0];
> eig(A)
ans =
  -0.3800 + 2.0990i
  -0.3800 - 2.0990i
```

They can also be obtained via:

```
> p=[1 .76 4.55];
> roots(p)
ans =
  -0.3800 + 2.0990i
  -0.3800 - 2.0990i
```

(c) The damping ratio, ς, and undamped natural frequency, ω_n.

Solution: These are easily computed: $\omega_n = 2.13\, r/s \quad ; \quad \varsigma = 0.178$

(d) The response of the plane to a 5° rudder input. Assume initial conditions are: $\Delta\beta(0) = \Delta r(0) = 0$.

Solution: A homework problem? □